

HARMONICITY OF UNIT VECTOR FIELDS WITH RESPECT TO A CLASS OF RIEMANNIAN METRICS

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ABSTRACT. The isotropic almost complex structures induce a Riemannian metric $g_{\delta,\sigma}$ on TM , which are the generalized type of Sasakian metric. In this paper, the Levi-Civita connection of $g_{\delta,\sigma}$ is calculated and the harmonicity of unit vector fields from (M, g) to $(S(M), i^*g_{\delta,0})$ is investigated, where $i^*g_{\delta,0}$ is a particular type of induced metric $i^*g_{\delta,\sigma}$. Finally, an important example is presented which satisfies in main theorem of the paper.

Keywords: Tangent bundle, unit tangent bundle, isotropic almost complex structure, energy functional, variational problem.

MSC(2010): 53C25, 53C40.

1. INTRODUCTION

Let (M, g) be a Riemannian manifold and (TM, g_s) be its tangent bundle equipped with Sasaki metric. Moreover, suppose $(S(M), i^*g_s)$ is the unit tangent bundle of (M, g) with induced Sasaki metric. Denote by ΓTM the set of all smooth vector fields on M .

Since, every vector field defines a map from (M, g) to (TM, g_s) , it is natural to investigate the harmonicity of vector fields as a map with the exception that the energy functional is restricted to the vector fields on M instead of all functions from (M, g) to (TM, g_s) . Medrano [2] investigated the harmonicity of a vector field from (M, g) to (TM, g_s) and he proved that the parallel vector fields are the only ones when (M, g) is a compact Riemannian manifold.

One can study the harmonicity of vector fields using by variational problem. This links the tension tensor field and critical points of the energy functional. Ishihara [3] calculated the tension tensor field of a vector field as a map from (M, g) to (TM, g_s) and represented another equivalent to the harmonicity of them. He showed that, the necessary and sufficient conditions for the harmonicity of a vector field is the vanishing of its Laplacian, i.e., $\Delta_g X = 0$.

On a compact Riemannian manifold (M, g) , the conditions $\nabla X = 0$ and $\Delta_g X = 0$ are equivalent for an arbitrary vector field, so Medrano and Ishihara present two different Equivalences for harmonicity of an arbitrary vector field.

We know that, if we restrict the energy functional to the unit vector fields, the vanishing of $\Delta_g X$ ensure the harmonicity of unit vector fields as a map from compact Riemannian manifold (M, g) to $(S(M), i^*g_s)$. However, this is a big condition for a unit vector field to be a harmonic vector field, therefore, it is natural to investigate the harmonicity of unit vector fields as a map from (M, g) to $(S(M), i^*g_s)$. Wiegink [?] demonstrated that a unit vector field X is a harmonic unit vector field if and only if $\Delta_g X = \|\nabla X\|^2 X$.

The contribution on the harmonicity of vector fields did not limited to the tangent bundles equipped with Sasaki metric. Abbassi and Calvaruso and Perrone [?] considered the problem of determining which vector fields $X : (M, g) \rightarrow (TM, G)$ define harmonic maps where G is an arbitrary g-natural metric on TM .

Aguilar [6] introduced Isotropic almost complex structures. Dragomir and Perrone [1] introduced the problem of studying the harmonic (unit) vector fields where TM equipped with Riemannian metric $g_{\delta, \sigma}$ induced by an arbitrary isotropic almost complex structure $J_{\delta, \sigma}$. In this paper we solve this problem when the unit tangent bundle $S(M)$ is equipped with induced Riemannian metric $i^*g_{\delta, 0}$, which is a particular type of $i^*g_{\delta, \sigma}$.

Section 2, gives a plenary preliminarie of the tangent bundle, unit tangent bundle, pullback tangent bundle, energy functional, variational problem and isotropic almost complex structures. Whereas Section 3 presents a discussion about the induced metrics $g_{\delta, \sigma}$ on TM and calculating of its Levi-civita connection. In section 4 we calculate the tension tensor field of an arbitrary vector field $X : (M, g) \rightarrow (TM, g_{\delta, \sigma})$ and section 5 is devoted to calculate the tension tensor field of unit vector fields from (M, g) to $(S(M), i^*g_{\delta, \sigma})$. In this section the critical points of the energy functional are determined.

2. PRELIMINARIES

2.1. Brief discussion about the tangent bundle and pullback tangent bundle. Let (M, g) be an n -dimensional Riemannian manifold and ∇ its Levi-Civita connection. Moreover let $\pi : TM \rightarrow M$ be its tangent bundle and $K : TTM \rightarrow TM$ be connection map with respect to ∇ . We can split TTM to vertical and horizontal

sub-vector bundles \mathcal{V} and \mathcal{H} , respectively, i.e., for every $v \in TM$, $T_v TM = \mathcal{V}_v \oplus \mathcal{H}_v$. These distributions have the following properties

- $\pi_{*v} |_{\mathcal{H}_v} : \mathcal{H}_v \longrightarrow T_{\pi(v)}M$ is an isomorphism.
- $K |_{\mathcal{V}_v} : \mathcal{V}_v \longrightarrow T_{\pi(v)}M$ is an isomorphism.

$X \in \Gamma(TM)$ has vertical and horizontal lifts $(X^v)_u = (K |_{\mathcal{V}_u})^{-1} X_{\pi(u)} \in \mathcal{V}_u$ and $(X^h)_u = (\pi_{*v} |_{\mathcal{H}_u})^{-1} X_{\pi(u)} \in \mathcal{H}_u$, when $X_{\pi(u)} \in T_{\pi(u)}M$.

Let $(p, u) \in TM$, the Lie bracket of the horizontal and vertical vector fields are expressed as follows

$$(2.1) \quad [X^h, Y^h]_{(p,u)} = [X, Y]_{(p,u)}^h - (R(X, Y)u)_{(p,u)}^v,$$

$$(2.2) \quad [X^h, Y^v]_{(p,u)} = (\nabla_X Y)_{(p,u)}^v,$$

$$(2.3) \quad [X^v, Y^v]_{(p,u)} = 0_{(p,u)}.$$

Let $f = (f_1, \dots, f_n) : (M, g) \longrightarrow (N, h)$ be a C^∞ function between Riemannian manifolds and ∇^M and ∇^N be Levi-Civita connections of g and h , respectively. If TN be the tangent bundle of N , then we can define vector bundle $f^{-1}TN$ on M by

$$(f^{-1}TN)_p = T_{f(p)}N \quad \forall p \in M.$$

There is a natural connection on $f^{-1}TN$ defined by

$$(f^{-1}\nabla^N)_{\frac{\partial}{\partial x^i}} Y_\beta = \frac{\partial f^\alpha}{\partial x^i} [(\Gamma^N)_{\alpha\beta}^\gamma of] Y_\gamma \quad \forall i = 1, \dots, m \quad \text{and} \quad \alpha, \beta, \gamma = 1, \dots, n,$$

where (x^1, \dots, x^m) be a local chart on M and (y^1, \dots, y^n) be a local chart on N and $\{Y_1 = \frac{\partial}{\partial y^1} of, \dots, Y_n = \frac{\partial}{\partial y^n} of\}$ is a local basis of sections on $f^{-1}TN$ and $\{(\Gamma^N)_{\alpha\beta}^\gamma\}$ are the Christoffel symbols of the Levi-Civita connection of h .

2.2. Energy functional on vector fields. Let (M, g) be an n -dimensional compact manifold and (TM, G) be its tangent bundle equipped with an arbitrary Riemannian metric G . Moreover, let $F : (M, g) \longrightarrow (TM, G)$ be an arbitrary smooth function. Then, the energy of F is defined by

$$E(F) = \frac{1}{2} \int_M \|dF\|^2 d\text{vol}(g),$$

where $\|dF\|$ is the Hilbert-Schmidt norm of dF , i.e., $\|dF\| = \text{tr}_g(F^*G)$ and $d\text{vol}(g)$ is the Riemannian volume form on M . We call E the energy functional.

In particular, we can restrict the energy functional to the sections of TM , that is, all of the vector fields on M . Due to this, we call E the energy functional on vector fields.

2.3. Variational problems on vector fields. Given an arbitrary tangent vector field $X \in \Gamma TM$. A smooth 1-parameter variation on X is a map $\mathcal{U} : M \times (-\delta, \delta) \rightarrow TM$ with this feature that \mathcal{U}_t is a vector field on M defined by $\mathcal{U}_t(p) = \mathcal{U}(p, t) \in T_p M$, for every $t \in (-\delta, \delta)$ and $\mathcal{U}(p, 0) = X_p$. We can define a vector field on $\mathcal{U}(M \times \{0\}) = X(M) \subseteq TM$ defined by $\mathcal{V}_{\mathcal{U}(p,0)} = \frac{d}{dt}|_{t=0} \mathcal{U}(p, t)$. It is clear that \mathcal{V} is a vertical vector field and we call it variational vector field.

Variational problem on vector fields calculates the critical points of E , hereafter called as harmonic vector fields.

2.4. Isotropic almost complex structures. Isotropic almost complex structures are a generalized type of natural almost complex structure on TM due to Aguilar [6]. The isotropic almost complex structures are determining an almost kahler metric whose kahler 2-form is the pullback of the canonical symplectic form on T^*M to TM via $\mathfrak{b} : TM \rightarrow T^*M$. Moreover, Aguilar [6] proved that there is an isotropic complex structure on M if and only if M is of constant sectional curvature.

Definition 2.1. [6] An almost complex structure J on TM is said to be isotropic with respect to the Riemannian metric on M , if there are smooth functions $\alpha, \delta, \sigma : TM \rightarrow R$ such that $\alpha\delta - \sigma^2 = 1$ and

$$(2.4) \quad JX^h = \alpha X^v + \sigma X^h, \quad JX^v = -\sigma X^v - \delta X^h.$$

3. A CLASS OF RIEMANNIAN METRICS ON TM

Let $\Theta \in \Omega^1(TM)$ be a 1-form on TM defined by

$$(3.1) \quad \Theta_v(A) = g_{\pi(v)}(\pi_*(A), v), \quad A \in T_v TM, \quad v \in TM.$$

Definition 3.1. [1] Let $J_{\delta, \sigma}$ be an isotropic almost complex structure. A $(0,2)$ -tensor $g_{\delta, \sigma}(A, B) = d\Theta(J_{\delta, \sigma}A, B)$ defines a Riemannian metric on TM provided that $0 \leq \alpha$ where $A, B \in \Gamma TTM$.

A simple calculation shows that

$$(3.2) \quad g_{\delta, \sigma}(X^h, Y^h) = \alpha g(X, Y) \circ \pi,$$

$$(3.3) \quad g_{\delta, \sigma}(X^h, Y^v) = -\sigma g(X, Y) \circ \pi,$$

$$(3.4) \quad g_{\delta, \sigma}(X^v, Y^v) = \delta g(X, Y) \circ \pi.$$

For calculating the Levi-Civita connection of $g_{\delta, \sigma}$ we need the following lemma.

Lemma 3.2. [7] *Let X, Y and Z be any vector field on M . Then*

$$(3.5) \quad X^h(g(Y, Z)o\pi) = (Xg(Y, Z))o\pi,$$

$$(3.6) \quad X^v(g(Y, Z)o\pi) = 0.$$

Theorem 3.3. *Let $g_{\delta, \sigma}$ be a Riemannian metric on TM as above. Then the Levi-Civita connection $\bar{\nabla}$ of $g_{\delta, \sigma}$ at $(p, u) \in TM$ is given by*

$$(3.7) \quad \begin{aligned} \bar{\nabla}_{X^h} Y^h &= (\nabla_X Y)^h - \frac{\sigma}{\alpha} (R(u, X)Y)^h + \frac{1}{2\alpha} X^h(\alpha)Y^h + \frac{1}{2\alpha} Y^h(\alpha)X^h \\ &\quad - \frac{\sigma}{\delta} (\nabla_X Y)^v - \frac{1}{2} (R(X, Y)u)^v - \frac{1}{2\delta} X^h(\sigma)Y^v \\ &\quad - \frac{1}{2\delta} Y^h(\sigma)X^v - \frac{1}{2} g(X, Y) \bar{\nabla} \alpha, \end{aligned}$$

$$(3.8) \quad \begin{aligned} \bar{\nabla}_{X^h} Y^v &= -\frac{\sigma}{\alpha} (\nabla_X Y)^h + \frac{\delta}{2\alpha} (R(u, Y)X)^h - \frac{1}{2\alpha} X^h(\sigma)Y^h + \frac{1}{2\alpha} Y^v(\alpha)X^h \\ &\quad + (\nabla_X Y)^v + \frac{1}{2\delta} X^h(\delta)Y^v - \frac{1}{2\delta} Y^v(\sigma)X^v + \frac{1}{2} g(X, Y) \bar{\nabla} \sigma, \end{aligned}$$

$$(3.9) \quad \begin{aligned} \bar{\nabla}_{X^v} Y^h &= \frac{\delta}{2\alpha} (R(u, X)Y)^h + \frac{1}{2\alpha} X^v(\alpha)Y^h - \frac{1}{2\alpha} Y^h(\sigma)X^h \\ &\quad - \frac{1}{2\delta} X^v(\sigma)Y^v + \frac{1}{2\delta} Y^h(\delta)X^v + \frac{1}{2} g(X, Y) \bar{\nabla} \sigma, \end{aligned}$$

$$(3.10) \quad \begin{aligned} \bar{\nabla}_{X^v} Y^v &= -\frac{1}{2\alpha} X^v(\sigma)Y^h - \frac{1}{2\alpha} Y^v(\sigma)X^h + \frac{1}{2\delta} X^v(\delta)Y^v + \frac{1}{2\delta} Y^v(\delta)X^v \\ &\quad - \frac{1}{2} g(X, Y) \bar{\nabla} \delta. \end{aligned}$$

Proof. We just prove (14), the remaining ones are similar. Using Koszul formula, we have

$$\begin{aligned} 2g_{\delta, \sigma}(\bar{\nabla}_{X^h} Y^h, Z^h) &= X^h g_{\delta, \sigma}(Y^h, Z^h) + Y^h g_{\delta, \sigma}(X^h, Z^h) - Z^h g_{\delta, \sigma}(X^h, Y^h) \\ &\quad + g_{\delta, \sigma}([X^h, Y^h], Z^h) + g_{\delta, \sigma}([Z^h, X^h], Y^h) \\ &\quad - g_{\delta, \sigma}([Y^h, Z^h], X^h). \end{aligned}$$

Using (1), (7) and (12) gives us

$$\begin{aligned} 2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^h) &= X^h(\alpha)g(Y, Z) + \alpha Xg(Y, Z) + Y^h(\alpha)g(X, Z) \\ &\quad + \alpha Yg(X, Z) - Z^h(\alpha)g(X, Y) - \alpha Zg(X, Y) + \alpha g([X, Y], Z) \\ &\quad + \sigma g(R(X, Y)u, Z) + \alpha g([Z, X]Y) + \sigma g(R(Z, X)u, Y) \\ &\quad - \alpha g([Y, Z], X) - \sigma g(R(Y, Z)u, X). \end{aligned}$$

Using the properties of the Levi-Civita connection of g and the compatibility of g with it, one can get

$$\begin{aligned} 2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^h) &= g(X^h(\alpha)Y, Z) + g(Y^h(\alpha)X, Z) - Z^h(\alpha)g(X, Y) \\ &\quad + 2\alpha g(\nabla_X Y, Z) + \sigma g(R(X, Y)u, Z) + \sigma g(R(Z, X)u, Y) \\ &\quad - \sigma g(R(Y, Z)u, X). \end{aligned}$$

Taking into account (7) and the Bianchi's first identity, we have

$$\begin{aligned} 2g_{\delta,\sigma}(\bar{\nabla}_{X^h}Y^h, Z^h) &= g_{\delta,\sigma}\left(\frac{1}{\alpha}X^h(\alpha)Y^h + \frac{1}{\alpha}Y^h(\alpha)X^h - g(X, Y)\bar{\nabla}\alpha + 2(\nabla_X Y)^h\right. \\ &\quad \left. - \frac{2\sigma}{\alpha}(R(u, X)Y)^h, Z^h\right), \end{aligned}$$

so the horizontal component of $\bar{\nabla}_{X^h}Y^h$ is

$$\begin{aligned} h(\bar{\nabla}_{X^h}Y^h) &= \frac{1}{2\alpha}X^h(\alpha)Y^h + \frac{1}{2\alpha}Y^h(\alpha)X^h - \frac{1}{2}g(X, Y)h(\bar{\nabla}\alpha) + (\nabla_X Y)^h \\ &\quad - \frac{\sigma}{\alpha}(R(u, X)Y)^h, \end{aligned}$$

where $\bar{\nabla}\alpha = h(\bar{\nabla}\alpha) + v(\bar{\nabla}\alpha)$ is the splitting of the gradient vector field of α with respect to $g_{\delta,\sigma}$ to horizontal and vertical components, respectively. Similarly the vertical component of $\bar{\nabla}_{X^h}Y^h$ is

$$\begin{aligned} v(\bar{\nabla}_{X^h}Y^h) &= -\frac{1}{2\delta}X^h(\sigma)Y^v - \frac{1}{2\delta}Y^h(\sigma)X^v - \frac{1}{2}g(X, Y)v(\bar{\nabla}\alpha) - \frac{\sigma}{\delta}(\nabla_X Y)^v \\ &\quad - \frac{1}{2}(R(X, Y)u)^v. \end{aligned}$$

Using the equation $(\bar{\nabla}_{X^h}Y^h) = h(\bar{\nabla}_{X^h}Y^h) + v(\bar{\nabla}_{X^h}Y^h)$, the proof is completed. \square

4. TENSION TENSOR FIELD OF $X : (M, g) \longrightarrow (TM, g_{\delta,\sigma})$

This section is devoted to calculate the tension tensor field of $X : (M, g) \longrightarrow (TM, g_{\delta,\sigma})$. The methods and the basic definitions and lemmas that we will use, can be found in [1].

According to definition, the tension tensor field of X is given by

$$(4.1) \quad \tau(X) = tr_g(\beta),$$

where $\beta \in \Gamma(T^*M \otimes T^*M \otimes X^{-1}TTM)$ defined by

$$(4.2) \quad \beta(Z, W) = (X^{-1}\bar{\nabla})_Z X_* W - X_*(\nabla_Z W) \quad Z, W \in \Gamma(TM),$$

the so called the second fundamental form of X .

For determining of the expression of the tension tensor field of X , we begin with some lemmas.

Lemma 4.1. [1] *let $V, X : M \longrightarrow TM$ be any smooth vector field. Then*

(4.3)

$$(\nabla_{\frac{\partial}{\partial x^i}} X)^k = \nabla_i \lambda^k = \frac{\partial \lambda^k}{\partial x^i} + N_i^k, \quad X_* V = V^h + (\nabla_V X)^v = V^i (\partial_i + \frac{\partial \lambda^j}{\partial x^i} \dot{\partial}_j),$$

where $(\nabla_{\frac{\partial}{\partial x^i}} X)^k$ is the k -th component of covariant derivative of $X = \lambda^j \frac{\partial}{\partial x^j}$ along $\frac{\partial}{\partial x^i}$ and X_* is the derivative of X as a map from M to TM .

with a simple calculation, we have

$$(4.4) \quad \bar{\nabla}_{V^h} V^h = V(V^j) \delta_j + V^i V^j \{ \bar{\nabla}_{\partial_i} \partial_j - N_i^k \bar{\nabla}_{\dot{\partial}_k} \partial_j - N_j^l \bar{\nabla}_{\partial_i} \dot{\partial}_l \}$$

$$(4.5) \quad + (N_i^k \Gamma_{jk}^l - \lambda^k \frac{\partial \Gamma_{jk}^l}{\partial x^i}) \dot{\partial}_l + N_i^k N_j^l \bar{\nabla}_{\dot{\partial}_k} \dot{\partial}_l \},$$

$$(4.6) \quad \bar{\nabla}_{V^h} (\nabla_V X)^v = V^i \frac{\partial}{\partial x^i} (\nabla_V X)^j \dot{\partial}_j + V^i (\nabla_V X)^j (\bar{\nabla}_{\partial_i} \dot{\partial}_j - N_i^k \bar{\nabla}_{\dot{\partial}_k} \dot{\partial}_j),$$

$$(4.7) \quad \bar{\nabla}_{(\nabla_V X)^v} V^h = V^i (\nabla_{\partial^i} X)^k V^j \{ \bar{\nabla}_{\dot{\partial}_k} \partial_j - \Gamma_{jk}^l \dot{\partial}_l - N_j^l \bar{\nabla}_{\dot{\partial}_k} \dot{\partial}_l \},$$

$$\bar{\nabla}_{(\nabla_V X)^v} (\nabla_V X)^v = (\nabla_V X)^k (\nabla_V X)^l \bar{\nabla}_{\dot{\partial}_k} \dot{\partial}_l,$$

equation (21) is because $\dot{\partial}_i (\nabla_V X)^k = 0$.

To calculate the tension tensor field of X , we need to define an important lemma specified as lemma 4.2.

Lemma 4.2. *Let $X = \lambda^i \frac{\partial}{\partial x^i}$ and $V = V^i \frac{\partial}{\partial x^i}$ be smooth vector fields on M . Then*

$$(4.8) \quad \begin{aligned} (X^{-1}\bar{\nabla})_V X_* V &= \bar{\nabla}_{V^h} V^h + \bar{\nabla}_{V^h} (\nabla_V X)^v \\ &+ \bar{\nabla}_{(\nabla_V X)^v} V^h + \bar{\nabla}_{(\nabla_V X)^v} (\nabla_V X)^v. \end{aligned}$$

Proof. Using (17) gives us

$$(X^{-1}\bar{\nabla})_V X_* V = (X^{-1}\bar{\nabla})_V V^i (\partial_i + \frac{\partial \lambda^j}{\partial x^i} \dot{\partial}_j).$$

According to the definition of covariant derivative, one can write

$$(X^{-1}\bar{\nabla})_V X_* V = V(V^i)(\partial_i + \frac{\partial \lambda^j}{\partial x^i} \dot{\partial}_j) + V^i V^j (X^{-1}\bar{\nabla})_{\frac{\partial}{\partial x^j}} (\partial_i + \frac{\partial \lambda^k}{\partial x^i} \dot{\partial}_k).$$

Using the definition of $X^{-1}\bar{\nabla}$ and the equation $\delta_i = \partial_i + N_i^j \dot{\partial}_j$, one can write

$$(4.9) \quad \begin{aligned} (X^{-1}\bar{\nabla})_V X_* V &= V(V^j)\delta_j + V^k \frac{\partial V^i}{\partial x^k} (N_i^j + \frac{\partial \lambda^j}{\partial x^i}) \dot{\partial}_j + V^i V^j \frac{\partial^2 \lambda^k}{\partial x^i \partial x^j} \dot{\partial}_k \\ &+ V^i V^j \{ \bar{\nabla}_{\partial_j} \partial_i + \frac{\partial \lambda^k}{\partial x^j} \bar{\nabla}_{\dot{\partial}_k} \partial_i + \frac{\partial \lambda^k}{\partial x^i} \bar{\nabla}_{\partial_j} \dot{\partial}_k + \frac{\partial \lambda^s}{\partial x^j} \frac{\partial \lambda^k}{\partial x^i} \bar{\nabla}_{\dot{\partial}_s} \dot{\partial}_k \}. \end{aligned}$$

With adding/subtracting $V^i V^j \frac{\partial}{\partial x^j} (\Gamma_{il}^k \lambda^l) \dot{\partial}_k$ in right hand side of equation (23), one can get

$$\begin{aligned} (X^{-1}\bar{\nabla})_V X_* V &= V(V^j)\delta_j + V^k \frac{\partial V^i}{\partial x^k} (\nabla_i \lambda^j) \dot{\partial}_j + V^i V^j \frac{\partial}{\partial x^j} (N_i^k + \frac{\partial \lambda^k}{\partial x^i}) \dot{\partial}_k \\ &+ V^i V^j \{ \bar{\nabla}_{\partial_i} \partial_j + (\nabla_j \lambda^k) \bar{\nabla}_{\dot{\partial}_k} \partial_i - N_j^k \bar{\nabla}_{\dot{\partial}_k} \partial_i + (\nabla_i \lambda^k) \bar{\nabla}_{\partial_j} \dot{\partial}_k \\ &- N_i^k \bar{\nabla}_{\partial_j} \dot{\partial}_k + \frac{\partial \lambda^s}{\partial x^j} \frac{\partial \lambda^k}{\partial x^i} \bar{\nabla}_{\dot{\partial}_s} \dot{\partial}_k \}, \end{aligned}$$

where we used the term $N_i^k = \Gamma_{il}^k \lambda^l$. Using the equation $\frac{\partial}{\partial x^j} (\Gamma_{il}^k \lambda^l) = \lambda^l \frac{\partial \Gamma_{il}^k}{\partial x^j} + \Gamma_{il}^k \frac{\partial \lambda^l}{\partial x^j}$ and (17), we have

$$\begin{aligned} (X^{-1}\bar{\nabla})_V X_* V &= V(V^j)\delta_j + V^i V^j \{ \bar{\nabla}_{\partial_i} \partial_j - N_j^k \bar{\nabla}_{\partial_i} \dot{\partial}_k - N_i^k \bar{\nabla}_{\dot{\partial}_k} \partial_j \} \\ &- V^i V^j \lambda^l \frac{\partial \Gamma_{il}^k}{\partial x^j} \dot{\partial}_k + V^i V^j \Gamma_{il}^k N_j^l \dot{\partial}_k - V^i V^j \Gamma_{il}^k (\nabla_j \lambda^l) \dot{\partial}_k \\ &+ V^j \frac{\partial V^i}{\partial x^j} (\nabla_i \lambda^k) \dot{\partial}_k + V^i V^j \frac{\partial}{\partial x^j} (\nabla_i \lambda^k) \dot{\partial}_k + V^i V^j \{ (\nabla_j \lambda^k) \bar{\nabla}_{\dot{\partial}_k} \partial_i \\ &(\nabla_i \lambda^k) \bar{\nabla}_{\partial_j} \dot{\partial}_k + \frac{\partial \lambda^s}{\partial x^j} \frac{\partial \lambda^k}{\partial x^i} \bar{\nabla}_{\dot{\partial}_s} \dot{\partial}_k \}. \end{aligned}$$

Substituting (18), (19), (20) and (21) in above equation completes the proof. \square

For calculating the tension tensor field of X , we need to calculate the second fundamental form of X at (V, V) .

$$\begin{aligned}
\beta_X(V, V) &= (X^{-1}\bar{\nabla})_V X_* V - X_*(\nabla_V V) \\
&= \frac{1}{\alpha} \{ (V^h(\alpha) + (\nabla_V X)^v(\alpha))V - \sigma R(X, V)V - (V^h(\sigma) \\
&\quad + (\nabla_V X)^v(\sigma))\nabla_V X - \sigma \nabla_V \nabla_V X + \delta R(X, \nabla_V X)V \}^h \\
&\quad + \frac{1}{\delta} \{ -(V^h(\sigma) + (\nabla_V X)^v(\sigma))V - \sigma \nabla_V V + (V^h(\delta) \\
&\quad + (\nabla_V X)^v(\delta))\nabla_V X + \delta \nabla_V \nabla_V X - \delta \nabla_{\nabla_V V} X \}^v \\
(4.10) \quad &- \frac{1}{2}g(V, V)\bar{\nabla}\alpha + g(V, \nabla_V X)\bar{\nabla}\sigma - \frac{1}{2}g(\nabla_V X, \nabla_V X)\bar{\nabla}\delta
\end{aligned}$$

where we used (11), (12), (13) and (14) in (22) and the equation $X_*(\nabla_V V) = (\nabla_V V)^h + (\nabla_{\nabla_V V} X)^v$.

So after long calculations we have the following theorem.

Theorem 4.3. *Let (M, g) be a Riemannian manifold, not necessarily compact, and $X \in \Gamma TM$ be a tangent vector field on M and $\{V_1, \dots, V_n\}$ be a locally orthonormal basis for TM . The tension tensor field of X , i.e., $\tau \in \Gamma X^{-1}TT(M)$, is given by*

$$\begin{aligned}
\tau(X) &= \\
&= \frac{1}{\alpha} \{ \pi_*(\bar{\nabla}\alpha) + \sum_{i=1}^n [(\nabla_{V_i} X)^v(\alpha))V_i] - \sigma Ric(X) - \sum_{i=1}^n [(V_i^h(\sigma) \\
&\quad + (\nabla_{V_i} X)^v(\sigma))\nabla_{V_i} X] - \sigma tr_g(\nabla \cdot \nabla \cdot X) + \delta tr_g R(X, \nabla \cdot X) \}^h \\
&\quad + \frac{1}{\delta} \{ - \sum_{i=1}^n [(V_i^h(\sigma) + (\nabla_{V_i} X)^v(\sigma))V_i] - \sigma \sum_{i=1}^n \nabla_{V_i} V_i \\
&\quad + \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta))\nabla_{V_i} X] + \delta \Delta_g X \}^v \\
(4.11) \quad &- \frac{n}{2}\bar{\nabla}\alpha + div(X)\bar{\nabla}\sigma - \frac{1}{2}g(\nabla X, \nabla X)\bar{\nabla}\delta
\end{aligned}$$

Proof. Using (15) gives us

$$\begin{aligned}
\tau(X) &= \sum_{i=1}^n \beta_X(V_i, V_i) \\
&= \sum_{i=1}^n \left\{ \frac{1}{\alpha} \{ (V_i^h(\alpha) + (\nabla_{V_i} X)^v(\alpha)) V_i - \sigma R(X, V_i) V_i - (V_i^h(\sigma) \right. \\
&\quad + (\nabla_{V_i} X)^v(\sigma)) \nabla_{V_i} X - \sigma \nabla_{V_i} \nabla_{V_i} X + \delta R(X, \nabla_{V_i} X) V_i \}^h \\
&\quad + \frac{1}{\delta} \{ -(V_i^h(\sigma) + (\nabla_{V_i} X)^v(\sigma)) V_i - \sigma \nabla_{V_i} V_i + (V_i^h(\delta) \\
&\quad + (\nabla_{V_i} X)^v(\delta)) \nabla_{V_i} X + \delta \nabla_{V_i} \nabla_{V_i} X - \delta \nabla_{\nabla_{V_i} V_i} X \}^v \\
(4.12) \quad &\left. - \frac{1}{2} g(V_i, V_i) \bar{\nabla} \alpha + g(V_i, \nabla_{V_i} X) \bar{\nabla} \sigma - \frac{1}{2} g(\nabla_{V_i} X, \nabla_{V_i} X) \bar{\nabla} \delta \right\}.
\end{aligned}$$

By substituting the following equations in (26) completes the proof.

$$\begin{aligned}
\sum_{i=1}^n V_i^h(\alpha) V_i^h &= \sum_{i=1}^n g_{\delta, \sigma}(\bar{\nabla} \alpha, V_i^h) V_i^h = h(\bar{\nabla} \alpha) \\
\sum_{i=1}^n R(X, V_i) V_i &= Ric(X) \\
\sum_{i=1}^n \nabla_{V_i} \nabla_{V_i} X &= tr_g \nabla \cdot \nabla X \\
\sum_{i=1}^n R(X, \nabla_{V_i} X) V_i &= tr_g R(X, \nabla X). \\
\sum_{i=1}^n [\nabla_{V_i} \nabla_{V_i} X - \nabla_{\nabla_{V_i} V_i} X] &= \Delta_g X
\end{aligned}$$

□

We end this section with a basic theorem in harmonic vector fields, which is very important to calculate the critical points of energy functional.

Theorem 4.4. *let (M, g) be a compact orientable Riemannian manifold and $E : \Gamma(TM) \rightarrow R^+$ the energy functional restricted to the space of all vector fields. Moreover, let $X_t : (-\varepsilon, +\varepsilon) \rightarrow TM$ be a variation along X and V be its variational vector field. If the tangent bundle of M is equipped with an arbitrary Riemannian metric G , Then*

$$(4.13) \quad \frac{d}{dt} \{E(X_t)\} \big|_{t=0} = - \int_M g_{\delta, \sigma}(V^v, \tau(X)) dvol(g),$$

where X is supposed as a map from (M, g) to (TM, G) .

5. TENSION TENSOR FIELD OF UNIT VECTOR FIELDS FROM (M, g) TO $(S(M), g_{\delta, \sigma})$ AND HARMONICITY OF THEM

The spherical bundle $S(M)$ on (M, g) at every point $p \in M$ is

$$S_p(M) = \{v \in T_p M | g(v, v) = 1\}.$$

The tangent space of $S(M)$ is $\mathcal{H} \oplus \bar{\mathcal{V}}$ where \mathcal{H} is the horizontal sub-bundle of TTM with respect to ∇ and $\bar{\mathcal{V}}$ is the vector bundle on $S(M)$ defined by

$$(5.1) \quad \bar{\mathcal{V}}_{(p,u)} = \{Y_p^v | g(Y_p, u) = 0\} = \{Y_p^v - g(Y_p, u)u^v | Y_p \in T_p M\} \quad (p, u) \in TM.$$

Assuming that $N_{(p,u)}^{g_{\delta, \sigma}} = \frac{\frac{\sigma}{\alpha}u^h + u^v}{\sqrt{\sigma - \frac{2\sigma^2}{\alpha} + \delta}}$ is a vector field on TM , one can simply derive $N_{(p,u)}^{g_{\delta, \sigma}}$ is normal unit vector field to $T_{(p,u)}S(M)$ with respect to $g_{\delta, \sigma}$.

We equip $S(M)$ with induced metric $i^*g_{\delta, \sigma}$, where $i : S(M) \rightarrow TM$ is the inclusion map. From [1], the tension tensor field of $X : (M, g) \rightarrow (S(M), i^*g_{\delta, \sigma})$ is the tangent part of $\tau(X)$ with respect to $g_{\delta, \sigma}$, i.e., $\tau_1(X) = \tan \tau(X)$. So, we have the following proposition.

Proposition 5.1. *Let $X : (M, g) \rightarrow (S(M), i^*g_{\delta, \sigma})$ be a unit vector field on M . Then the tension tensor field of X is*

$$(5.2) \quad \tau_1(X) = \tau(X) - g_{\delta, \sigma}(\tau(X), \frac{\frac{\sigma}{\alpha}X^h + X^v}{\sqrt{\sigma - \frac{2\sigma^2}{\alpha} + \delta}}) \frac{\frac{\sigma}{\alpha}X^h + X^v}{\sqrt{\sigma - \frac{2\sigma^2}{\alpha} + \delta}}.$$

Proof. According to the definition of tension tensor field of X , $\tau_1(X)$ is tangent to spherical bundle at X , i.e., $\tau_1(X) \in T_X S(M)$. Since $\frac{\frac{\sigma}{\alpha}X^h + X^v}{\sqrt{\sigma - \frac{2\sigma^2}{\alpha} + \delta}}$ is normal unit vector to $S(M)$ at X , we have

$$\tau_1(X) = \tau(X) - g_{\delta, \sigma}(\tau(X), \frac{\frac{\sigma}{\alpha}X^h + X^v}{\sqrt{\sigma - \frac{2\sigma^2}{\alpha} + \delta}}) \frac{\frac{\sigma}{\alpha}X^h + X^v}{\sqrt{\sigma - \frac{2\sigma^2}{\alpha} + \delta}},$$

and the proof is completed. \square

Hereafter, we assume that $\sigma = 0$, and present the essential and sufficient conditions for harmonicity of unit vector field $X : (M, g) \rightarrow (S(M), i^*g_{\delta, 0})$.

Theorem 5.2. *If the horizontal and vertical sub-bundles are perpendicular to each other, i.e., $\sigma = 0$, then the tension tensor field of unit vector field X is given by*

$$\begin{aligned}
 \tau_1(X) = & \frac{1}{\alpha} \{ \pi_*(\bar{\nabla}\alpha) + \sum_{i=1}^n [(\nabla_{V_i} X)^v(\alpha))V_i] + \delta tr_g R(X, \nabla X) \}^h \\
 & + \frac{1}{\delta} \{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta))\nabla_{V_i} X] + \delta \Delta_g X \\
 & - [\delta g(\Delta_g X, X) - \frac{n}{2} d\alpha(X^v) - \frac{1}{2} g(\nabla X, \nabla X) d\delta(X^v)] X \}^v \\
 (5.3) \quad & - \frac{n}{2} \bar{\nabla}\alpha - \frac{1}{2} g(\nabla X, \nabla X) \bar{\nabla}\delta,
 \end{aligned}$$

where $\Delta_g X$ is the Laplacian [1] of X defined by

$$-\Delta_g X = \sum_{i=1}^n [\nabla_{V_i} \nabla_{V_i} X - \nabla_{\nabla_{V_i} V_i} X],$$

and $\{V_i\}_{i=1}^n$ is a locally orthonormal basis for TM .

Proof. Substituting $\sigma = 0$ in (25) gives us

$$\begin{aligned}
 \tau(X) = & \frac{1}{\alpha} \{ \pi_*(\bar{\nabla}\alpha) + \sum_{i=1}^n [(\nabla_{V_i} X)^v(\alpha))V_i] + \delta tr_g R(X, \nabla X) \}^h \\
 & + \frac{1}{\delta} \{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta))\nabla_{V_i} X] + \delta \Delta_g X \}^v \\
 (5.4) \quad & - \frac{n}{2} \bar{\nabla}\alpha - \frac{1}{2} g(\nabla X, \nabla X) \bar{\nabla}\delta.
 \end{aligned}$$

Taking into account (31) and $\sigma = 0$ in (29) gives us

$$\begin{aligned}
 \tau_1(X) = & \frac{1}{\alpha} \{ \pi_*(\bar{\nabla}\alpha) + \sum_{i=1}^n [(\nabla_{V_i} X)^v(\alpha))V_i] + \delta tr_g R(X, \nabla X) \}^h \\
 & + \frac{1}{\delta} \{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta))\nabla_{V_i} X] + \delta \Delta_g X \}^v \\
 & - g_{\delta,0} \left(\frac{1}{\delta} \{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta))\nabla_{V_i} X] + \delta \Delta_g X \}^v \right. \\
 & \left. - \frac{n}{2} \bar{\nabla}\alpha - \frac{1}{2} g(\nabla X, \nabla X) \bar{\nabla}\delta, X^v \right) \frac{X^v}{\delta} - \frac{n}{2} \bar{\nabla}\alpha - \frac{1}{2} g(\nabla X, \nabla X) \bar{\nabla}\delta.
 \end{aligned}$$

Using the definition of $g_{\delta,0}$ and the fact that $g(\nabla_{V_i}X, X) = 0$ for every $i = 1, \dots, n$, we have

$$\begin{aligned}\tau_1(X) &= \frac{1}{\alpha} \{ \pi_*(\bar{\nabla}\alpha) + \sum_{i=1}^n [(\nabla_{V_i}X)^v(\alpha))V_i] + \delta tr_g R(X, \nabla X) \}^h \\ &\quad + \frac{1}{\delta} \{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i}X)^v(\delta))\nabla_{V_i}X] + \delta \Delta_g X \}^v \\ &\quad - \frac{1}{\delta} [g(\delta \Delta_g X, X) - \frac{n}{2} d\alpha(X^v) - \frac{1}{2} g(\nabla X, \nabla X) d\delta(X^v)] X^v \\ &\quad - \frac{n}{2} \bar{\nabla}\alpha - \frac{1}{2} g(\nabla X, \nabla X) \bar{\nabla}\delta,\end{aligned}$$

and the proof is completed. \square

From [1], we know that a map between Riemannian manifolds is harmonic if and only if the tension tensor field of it, is zero. So, the vector field $X : (M, g) \longrightarrow (S(M), i^*g_{\delta,0})$ is a harmonic map if and only if

$$\begin{aligned}&\frac{1}{\alpha} \{ \pi_*(\bar{\nabla}\alpha) + \sum_{i=1}^n [(\nabla_{V_i}X)^v(\alpha))V_i] + \delta tr_g R(X, \nabla X) \} \\ &\quad - \pi_*(\frac{n}{2} \bar{\nabla}\alpha + \frac{1}{2} g(\nabla X, \nabla X) \bar{\nabla}\delta) = 0,\end{aligned}$$

and

$$\begin{aligned}&\frac{1}{\delta} \{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i}X)^v(\delta))\nabla_{V_i}X] + \delta \Delta_g X - g(\delta \Delta_g X, X) \\ &\quad + \frac{n}{2} d\alpha(X^v) + \frac{1}{2} g(\nabla X, \nabla X) d\delta(X^v)] X \} - K(\frac{n}{2} \bar{\nabla}\alpha + \frac{1}{2} g(\nabla X, \nabla X) \bar{\nabla}\delta) = 0.\end{aligned}$$

According to a theorem [1] in harmonic theory on vector fields, if G is an arbitrary Riemannian metric on $S(M)$ and V is an orthogonal vector field to $X : (M, g) \longrightarrow (S(M), G)$ with respect to g , one can get

$$(5.5) \quad \int_M G(\tau_1(X), V^v) dvol(g) = 0.$$

Relation (32) will be used in proof of main theorem. Moreover, if \mathcal{V} is variational vector field of an arbitrary variation of X through unit vector fields, then $g(K(\mathcal{V}), X) = 0$.

The following lemma is a famous lemma in harmonic theory.

Lemma 5.3. *A unit vector field $X : (M, g) \longrightarrow (S(M), G)$ is harmonic if and only if*

$$(5.6) \quad \frac{d}{dt}\{E(\mathcal{U}_t)\} \big|_{t=0} = - \int_M G(V^v, \tau_1(X)) \, d\text{vol}(g) = 0,$$

where \mathcal{U}_t is a variation along X through unit vector fields and $\mathcal{V} = V^v$ is its variation vector field.

Main theorem of paper, specified as theorem 5.4. It presents condition on vector fields to be harmonic as a map from (M, g) to $(S(M), g_{\delta,0})$.

Theorem 5.4. *Let (M, g) be a compact orientable Riemannian manifold and X be a unit vector field on M . Then $X : (M, g) \longrightarrow (S(M), g_{\delta,0})$ is a harmonic vector field if and only if*

$$(5.7) \quad \begin{aligned} \Delta_g X &= \frac{1}{\delta}[(\delta - \frac{1}{2}d\delta(X^v))\|\nabla X\|^2 - \frac{n}{2}d\alpha(X^v)]X \\ &\quad + \frac{n}{2}K(\bar{\nabla}\alpha) + \frac{1}{2}\|\nabla X\|^2 K(\bar{\nabla}\delta) \\ &\quad - \frac{1}{\delta} \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta))\nabla_{V_i} X], \end{aligned}$$

where K is connection map with respect to the Levi-civita connection of g .

Proof. (\implies) Let X be a harmonic vector field, we show that (34) holds. suppose,

$$(5.8) \quad \tau_1(X) = \zeta X^h + \lambda X^v + V^v + W^h,$$

where V and W are perpendicular vector fields to X and $\lambda, \zeta : X(M) \longrightarrow R$ are C^∞ functions. we show that $V = 0$ and $\lambda = 0$. From (35), we have

$$(5.9) \quad \|V^v\|^2 = g_{\delta,0}(\tau_1(X), V^v).$$

According to (32), $\int_M \|V^v\|^2 d\text{vol}(g) = \int_M g_{\delta,0}(\tau_1(X), V^v) d\text{vol}(g) = 0$. This shows that $V = 0$, and $\tau_1(X) = \tau(X) - g_{\delta,0}(\tau(X), X^v) \frac{X^v}{\delta}$ shows that $\tau_1(X)$ hasn't any component in direction of X^v , i.e., $\lambda = 0$. From (35) and $V = 0$ and $\lambda = 0$, one can get $K(\tau_1(X)) = 0$. On the other

hand, from (30), we have

$$\begin{aligned}
 K(\tau_1(X)) &= \frac{1}{\delta} \left\{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta)) \nabla_{V_i} X] + \delta \Delta_g X \right\} \\
 &\quad - \frac{1}{\delta} [g(\delta \Delta_g X, X) - \frac{n}{2} d\alpha(X^v) - \frac{1}{2} g(\nabla X, \nabla X) d\delta(X^v)] X \\
 (5.10) \quad &\quad - \frac{n}{2} K(\bar{\nabla} \alpha) - \frac{1}{2} g(\nabla X, \nabla X) K(\bar{\nabla} \delta).
 \end{aligned}$$

Using $g(\Delta_g X, X) = \frac{1}{2} \Delta(\|X\|^2) + \|\nabla X\|^2$, we get

$$\begin{aligned}
 K(\tau_1(X)) &= \frac{1}{\delta} \left\{ \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta)) \nabla_{V_i} X] + \delta \Delta_g X \right\} \\
 &\quad - \frac{1}{\delta} [(\delta - \frac{1}{2} d\delta(X^v)) \|\nabla X\|^2 - \frac{n}{2} d\alpha(X^v)] X \\
 (5.11) \quad &\quad - \frac{n}{2} K(\bar{\nabla} \alpha) - \frac{1}{2} g(\nabla X, \nabla X) K(\bar{\nabla} \delta).
 \end{aligned}$$

Using $K(\tau_1(X)) = 0$ in (38) gives us

$$\begin{aligned}
 \Delta_g X &= \frac{1}{\delta} [(\delta - \frac{1}{2} d\delta(X^v)) \|\nabla X\|^2 - \frac{n}{2} d\alpha(X^v)] X \\
 &\quad + \frac{n}{2} K(\bar{\nabla} \alpha) + \frac{1}{2} g(\nabla X, \nabla X) K(\bar{\nabla} \delta) \\
 (5.12) \quad &\quad - \frac{1}{\delta} \sum_{i=1}^n [(V_i^h(\delta) + (\nabla_{V_i} X)^v(\delta)) \nabla_{V_i} X].
 \end{aligned}$$

(\Leftarrow) Let (34) holds, we show that X is a harmonic vector field. Substitute (34) in (37) gives us, $K(\tau_1(X)) = 0$, i.e., the vertical part of $\tau_1(X)$ is zero. Lemma 5.3. with $K(\tau_1(X)) = 0$ give

$$(5.13) \quad \frac{d}{dt} \{E(\mathcal{U}_t)\} |_{t=0} = - \int_M g_{\delta,0}(V^v, \tau_1(X)) d\text{vol}(g) = 0,$$

where \mathcal{U}_t is a variation along X through unit vector fields and $\mathcal{V} = V^v$ is its variation vector field. Note that the vertical and horizontal sub-bundles are perpendicular to each other with respect to $g_{\delta,0}$. \square

Corollary 5.5. *Theorem 5.4. includes the particular theorem of Wiegmann [4].*

Proof. Putting $\delta = 1$ in Theorem 5.4. proves the corollary. \square

The following proposition states that, if a unit vector field satisfies in (34) then it is a critical point of the following functional.

Proposition 5.6. *The energy functional $E : \Gamma(S(M), g_{\delta,0}) \longrightarrow R^+$ is given by*

$$(5.14) \quad E(X) = \frac{1}{2} \int_M (n\alpha + \delta \|\nabla X\|^2) d\text{vol}(g),$$

for every, unit vector field $X : (M, g) \longrightarrow (S(M), g_{\delta,0})$.

Proof. We know that the energy functional is given by

$$(5.15) \quad E(X) = \frac{1}{2} \int_M \|dX\|^2 d\text{vol}(g).$$

On the other hand

$$\begin{aligned} \|dX\|^2 &= \text{tr}_g X^* g_{\delta,0} = \sum_{i=1}^n g_{\delta,0}(X_* V_i, X_* V_i) \\ &= \sum_{i=1}^n g_{\delta,0}(V_i^h + (\nabla_{V_i} X)^v, V_i^h + (\nabla_{V_i} X)^v) \\ (5.16) \quad &= n\alpha + \delta \|\nabla X\|^2, \end{aligned}$$

where $\{V_1, \dots, V_n\}$ is an orthonormal locally basis for TM . \square

We end our work with some examples on harmonic vector fields satisfying Theorem 5.4.

Example 5.7. Let $(R^n, \langle \cdot, \cdot \rangle)$ be Euclidean space and (TR^n, g_s) be its tangent bundle equipped with Sasaki metric. Then, every parallel unit vector field is harmonic unit vector field. In this example, isotropic almost complex structure is the natural almost complex structure on TR^n .

Example 5.8. Define three complex structures J_1, J_2 and J_3 on Euclidean four space $(E^4, \langle \cdot, \cdot \rangle)$ by

$$(5.17) \quad J_1(v_1, v_2, v_3, v_4) = (-v_2, v_1, -v_4, v_3),$$

$$(5.18) \quad J_2(v_1, v_2, v_3, v_4) = (v_3, -v_4, -v_1, v_2),$$

$$(5.19) \quad J_3(v_1, v_2, v_3, v_4) = (v_4, v_3, -v_2, -v_1),$$

where (v_1, v_2, v_3, v_4) is a tangent vector to R^4 . It is simple to check that $(R^4, J_i, \langle \cdot, \cdot \rangle)$ for $i = 1, 2, 3$ are kahler manifolds. Moreover, suppose $p = (p_1, p_2, p_3, p_4)$ be a point in S^3 and $N(p) = (p_1, p_2, p_3, p_4)$ be position vector field. Let $X_1(p) = J_1 N$, $X_2(p) = J_2 N$ and $X_3(p) = J_3 N$ be tangent vector fields on S^3 . One can check that X_1, X_2 and X_3 are unit vector fields and are perpendicular to each other, i.e., $\{X_1, X_2, X_3\}$ is an orthonormal basis for TS^3 .

We will show that $X_1 : (S^3, g) \longrightarrow (S(S^3), g_{\delta,0})$ is a harmonic unit vector field, That is, we will show that X_1 satisfies in (34) (g is induced by Euclidean metric on S^3).

Using Gauss formula gives us the Levi-Civita connection ∇^S of g as following

$$(5.20) \quad \nabla_Z^S W = \nabla_Z^E W + g(Z, W)N.$$

If V is a vector field on S^3 then from (47) one can get

$$(5.21) \quad \begin{aligned} \nabla_V^S X_1 &= \nabla_V^E X_1 + g(V, X_1)N = \nabla_V^E J_1 N + g(V, X_1)N \\ &= J_1 V + g(V, X_1)N. \end{aligned}$$

By the definition, the Laplacian of X_1 is

$$(5.22) \quad \begin{aligned} -\Delta_g X_1 &= \nabla_{X_1}^S \nabla_{X_1}^S X_1 + \nabla_{X_2}^S \nabla_{X_2}^S X_1 + \nabla_{X_3}^S \nabla_{X_3}^S X_1 \\ &\quad - \sum_{i=1}^3 \nabla_{\nabla_{X_i}^S X_i}^S X_1. \end{aligned}$$

Using by (48) and (47) and the fact that $\nabla_{X_i}^S X_i = 0$ (because the integral curves of X_i for all $i = 1, 2, 3$ are geodesics of S^3), we have

$$(5.23) \quad -\Delta_g X_1 = J_1(\nabla_{X_2}^E X_2 + \nabla_{X_3}^E X_3).$$

With an straight forward calculation on (50), we have

$$(5.24) \quad -\Delta_g X_1 = -2J_1 N = -2X_1,$$

so, $\Delta_g X_1 = 2X_1$.

For calculating the right hand side of (34), we need to define $\alpha, \beta : TS^3 \longrightarrow R$. Let $V \in \Gamma(TS^3)$ be an arbitrary vector field on S^3 . Since, $X_i, i = 1, 2, 3$ are global sections and basis on TS^3 , we can write $V = \sum_{i=1}^3 V^i X_i$ where $V^i : TS^3 \longrightarrow R$ are smooth maps for $i = 1, 2, 3$. We define $\alpha, \beta : TS^3 \longrightarrow R$ with definitions $\alpha(V) = \frac{(V^1)^2}{2} + 1$ and $\beta(V) = \frac{1}{\alpha(V)}$. One can get

$$(5.25)$$

$$d\alpha(A) = V^1 dV^1(A) \quad \text{and} \quad (\bar{\nabla}\alpha)_V = V^1(\bar{\nabla}V^1)_V = \frac{V^1}{\delta}(X_1^v)_V,$$

$$(5.26)$$

$$d\delta(A) = -\delta^2 V^1 dV^1(A) \quad \text{and} \quad (\bar{\nabla}\delta)_V = -\delta V^1(X_1^v)_V.$$

Note that $\bar{\nabla}V^1 = \frac{X_1^v}{\delta}$. Since in the right hand side of (34), $d\delta$ and $d\alpha$ are calculated at X_1 and also the vlues of α and δ are calculated at X_1 ,

one can write

$$(5.27) \quad d\alpha|_{X_1}(X_1^v) = 1 \quad \text{and} \quad (\bar{\nabla}\alpha)_{X_1} = \left(\frac{X_1^v}{\delta}\right)_{X_1},$$

$$(5.28) \quad d\delta|_{X_1}(X_1^v) = -\delta^2 \quad \text{and} \quad (\bar{\nabla}\delta)_{X_1} = (-\delta X_1^v)_{X_1}.$$

Using (54) and (55), the right hand side of (34) is calculated as following

$$(5.29) \quad \begin{aligned} & \frac{1}{\delta}[(\delta + \frac{1}{2}\delta^2)2 - \frac{n}{2}]X_1 + \frac{n}{2\delta}X_1 - \delta X_1 \\ & = 2X_1. \end{aligned}$$

Note that the third line of (34) is zero. From (56) and (51) and (34) we conclude that X_1 is a harmonic unit vector field on S^3 .

6. CONCLUSIOS

In this paper, the harmonicity of unit vector fields were investigated, whereas the tangent bundle was equipped with a Riemannian metric which is induced by isotropic almost complex structures, for this reason, we need to calculate the Levi-Civita connection of itroduced metric and tension tensor field of unit vector fields. The proof of Lemma 4.2 was an important step for calculating the tension tensor field of vector fields.

In the Section 5, we calculated the tension tensor field of unit vector fields using by a helpful theorem which makes shortening the tension tensor field calculation. Since, For proving the main theorem we used the variational problem, Lemma 5.3 was most fundamental lemma to state.

Finally we present some examples satisfying in the main theorem. The first example is a particular example, because we used the Sasaki metric on the tangent bundle and the second example is an important example, because of its connection with Hopf vector fields. This example leads us to investigate the harmonicity of Hopf vector fields in this sense. Future work is checking the harmonicity of Hopf vector fields on 3-sphere.

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